On D. I. Moldavanskii's question about p-separable subgroups of a free group

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Abstract

We prove that every nonabelian free group has a finitely generated isolated subgroup which is not separable in the class of nilpotent groups.

This enables us to give a negative answer to the following question by D. I. Moldavanskii in the "Kourovka Notebook": Is it true that any finitely generated p'–isolated subgroup of a free group is separable in the class of finite p–groups?
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According to A. I. Mal'cev [1], a subgroup H of a group G is said to be *finitely separable from an element* $g \in G \setminus H$ if there is a homomorphism φ of G to some finite group such that $\varphi(g) \notin \varphi(H)$.

If a subgroup H of a given group is finitely separable from every element of this group which is not in H we will say that H is *finitely separable*. If we will consider homomorphisms not on a finite group but on groups from some class $\mathcal{K}$ we will define *separability in the class* $\mathcal{K}$. In particular, if $\mathcal{K}$ is the class of finite p–groups then we will say that a subgroup H is *p–separable*. In this article p is a fixed prime number. The problem of finite separability is closely related to the generalized word problem (see [1]). The *generalized word problem* for H in G asks for an algorithm that decides whether or not the elements of G lie in H.

M. Hall proved in [2] that every finitely generated subgroup of a free group is finitely separable. One easily sees that an analogue of Hall's theorem is not true for the class of all finite p–groups. Indeed, let $G = \langle a \rangle$ be an infinite cyclic group and $H = \langle a^q \rangle$ be a subgroup of G, where $q \neq p$ is a prime number. Evidently, $a \notin H$ but for every homomorphism from G onto a p–group the image of a lies in the image of H. Therefore, H is not p–separable.

In the previous example H is not p'–isolated in G. Recall that a subgroup H of G is called *p'–isolated*, if for every prime $q \neq p$ and for every $g \in G$ an inclusion $g^q \in H$ implies that $g \in H$. If in the previous example we will consider only p'–isolated subgroups, then each of them will be p–separable.

D. I. Moldavanskii suggested that the last fact is true in every free group.

Problem ([3, Problem 15.60]). *Is it true that any finitely generated p'–isolated subgroup of a free group is separable in the class of finite p–groups? It is easy to see that this is true for cyclic subgroups.*

E. D. Loginova [4, §3] proves that in each finitely generated nilpotent group every p'–isolated subgroup is p–separable. This theorem says that the Moldavanskii's hypothesis may be true.

In the present article we prove the following

Theorem. *In every free nonabelian group there is a finitely generated isolated subgroup which is not separable in the class of nilpotent groups.*

Recall that every finite p–group is nilpotent [5, p. 115] and every isolated subgroup is p'–isolated for every prime number p. Therefore, we obtain as consequence of the Theorem that D. I. Moldavanskii's conjecture is not true.

Corollary. *Let p be a prime number. In every free nonabelian group there is a finitely generated p'–isolated subgroup which is not separable in the class of finite p–groups.*

In now we can start to prove the Theorem. Consider a free nonabelian group

$$F = \langle x, y, z_i (i \in I) \rangle,$$

where I is some index set (possibly empty). Let H be a subgroup of G:

$$H = \langle x\,[y,\,x],\ y,\ z_j (j \in J) \rangle,$$

where $[y,\,x] = y^{-1}\,x^{-1}\,y\,x$ and J is a subset of I. We see that H is a proper subgroup of F and, in particular, the element x does not lie in H. Indeed, let $a = x\,[y,\,x]$, $b = y$. Obviously, every nonempty word, reduced over the alphabet $A = \{a^{\pm 1},\ b^{\pm 1},\ z_j^{\pm 1} (j \in J)\}$, is reduced over the alphabet $X = \{x^{\pm 1},\ y^{\pm 1},\ z_i^{\pm 1} (i \in I)\}$ and contains the letter y or y^{-1}. Therefore, none of these words coincide with x.

Let us prove now that the subgroup H is not separable in the class of nilpotent groups. To do this we consider the quotient groups $F/\gamma_n F$, where

$$\gamma_1 F = F, \quad \gamma_{i+1} F = [\gamma_i F, F], \quad i = 1, 2, \ldots,$$

are the terms of the lower central series of F. We also consider the natural homomorphisms

$$\varphi_n : F \longrightarrow F/\gamma_n F, \quad n = 1, 2, \ldots.$$

Note that for each φ_n the image $\varphi_n(H)$ is equal to the image of the subgroup $\langle x,\, y,\, z_j (j \in J) \rangle$. The statement is obvious for $n = 1, 2$. For the case when $n > 2$ we apply the following well-known fact [6, Theorem 31.2.5]: If a nilpotent group G is generated modulo the commutator subgroup $G' = \gamma_2 G$ by elements $a_1, a_2, \ldots, a_r$, then the elements $a_1, a_2, \ldots, a_r$ also generate G. Since all $F/\gamma_n F$ are free nilpotent groups, it follows that H is not separable in the class of all free nilpotent groups and hence H is not separable in the class of all nilpotent groups.

To complete the proof of the Theorem we must prove that H is isolated in F. For simplicity sake, we consider the case in which F does not include the generators z_i. The proof is similar in the general case.

Lemma. *The subgroup $H = \langle x\,[y,\,x],\ y\rangle$ is isolated in the free group $F = \langle x,\ y\rangle$.*

Proof. Let $a = x\,[y,\,x]$, $b = y$. We will consider the elements of F as words over the alphabet $X = \{x^{\pm 1}, y^{\pm 1}\}$. The elements of H will be considered both as words over the alphabet $A = \{a^{\pm 1}, b^{\pm 1}\}$ and the as words over the alphabet X on substituting for a and b their expressions over the alphabet X. By " $\equiv$ " will denote the graphical equality of words over the alphabet X, by " $=$ " will denote equality in F.

Let for some reduced word f from F and for some integer number m element f^m lies in H. Obviously we may consider only the case $m > 0$. Then for some integers $\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_k, \beta_k$, which are all nonzero except for, possibly, α_1 and β_k we have that

$$f^m = a^{\alpha_1}\, b^{\beta_1}\, a^{\alpha_2}\, b^{\beta_2}\, \ldots\, a^{\alpha_k}\, b^{\beta_k}. \tag{1}$$

Note that if the word on the right–hand side of this equality is reduced over the alphabet A, then it also reduced over the alphabet X. We will show that the reduced word f^m might be considered as cyclically reduced.

Indeed, consider the last letter of f^m. If it equals y, then $\beta_k > 0$. If the first letter of f^m is not y^{-1}, then the word f^m is cyclically reduced. If the first letter of f^m is y^{-1} then $\alpha_1 = 0$ and $\beta_1 < 0$. Therefore, the word on the right–hand side of (1) is not cyclically reduced over A. Conjugate both sides of (1) by the element $y^{-1} = b^{-1}$. We obtain an element in H which is the mth power of an element in F. The case when the last letter of f^m is y^{-1} is considered in a similar way.

Let x be the last letter of f^m. Then $\beta_k = 0$, $\alpha_k > 0$. If the first letter of f^m is not x^{-1}, then f^m is cyclically reduced. If the first letter of f^m is x^{-1}, then $\alpha_1 < 0$. Conjugating both sides of (1) by a^{-1}, we again find an element in H which is the mth power of some element in F. In the case in which the last letter of f^m is x^{-1} the argument is similar. Repeating this process, we will arrive to an equality of the same form as (1) where the word on the left–hand side is cyclically reduced, as desired.

So, we are assuming from now on that the word f^m on the left–hand side of (1) is cyclically reduced. In this case $f^m \equiv f_1\, f_2\, \ldots\, f_m$, where $f_i \equiv f$, i.e., there are no reductions in the products $f_i\, f_{i+1}$, $i = 1, 2, \ldots, m-1$. Consider the equality

$$f_1\, f_2\, \ldots\, f_m = a^{\alpha_1}\, b^{\beta_1}\, a^{\alpha_2}\, b^{\beta_2}\, \ldots\, a^{\alpha_k}\, b^{\beta_k}. \tag{2}$$

If f_1 is equal to an initial subword of the word from the right–hand side of (2) then f_1 is equal to some word over the alphabet A, i.e., $f = f_1 \in H$ and the statement is true.

We will prove that other possibilities are impossible.

Suppose that f_1 is a product of an initial subword on the right–hand side of (2) and some subword of a or a^{-1}, i.e., for some exponents α_l, $1 \le l \le k$, we have $a^{\alpha_l} \equiv a^{\gamma}\, a_0\, a_1\, a^{\delta}$, $|\alpha_l| = |\gamma| + |\delta| + 1$, and

$$f_1 \equiv a^{\alpha_1}\, b^{\beta_1}\, a^{\alpha_2}\, b^{\beta_2}\, \ldots\, b^{\beta_{l-1}}\, a^{\gamma}\, a_0,$$

where $a \equiv a_0\, a_1$ if $\alpha_l > 0$ and $a^{-1} \equiv a_0\, a_1$ if $\alpha_l < 0$. The first letter of f_1 may be one of the set $\{x^{\pm 1},\ \ y^{\pm 1}\}$. Consider all these possibilities.

Let the first letter of f_1 be x. Since $f_1 \equiv f_2$ it follows that f_2 must begin with x as well and, moreover, either

$$f_1 \equiv a^{\alpha_1}\, b^{\beta_1}\, a^{\alpha_2}\, b^{\beta_2}\, \ldots\, b^{\beta_{l-1}}\, a^{\gamma}\, x\, y^{-1}\, x^{-1}\, y, \quad \gamma \ge 0, \quad a_0 \equiv x\, y^{-1}\, x^{-1}\, y,$$

or

$$f_1 \equiv a^{\alpha_1}\, b^{\beta_1}\, a^{\alpha_2}\, b^{\beta_2}\, \ldots\, b^{\beta_{l-1}}\, a^{\gamma}\, x^{-1}\, y^{-1}, \quad \gamma \leq 0, \quad a_0 \equiv x^{-1}\, y^{-1}.$$

In the first case we have

$$f_2 \equiv x\, a^{\delta}\, b^{\beta_l}\, a^{\alpha_{l+1}}\, w,$$

where w is some reduced word. Since the first letter of f_1 is x, we have $\alpha_1 > 0$ and

$$f_1 \equiv x\, y^{-1}\, x^{-1}\, y\, x\, a^{\alpha_1 - 1}\, b^{\beta_1}\, \ldots\, a^{\gamma}\, x\, y^{-1}\, x^{-1}\, y.$$

In order to the first three letters of f_2 are equal to the first three letters of f_1 it is necessary that $\delta = 0$, $\beta_l = -1$, $\alpha_{l+1} < 0$. But in this case

$$f_2 \equiv x\, y^{-1}\, x^{-1}\, y^{-1}\, x\, y\, x^{-1}\, a^{\alpha_{l+1}+1}\, w.$$

By comparing the fourth letters of f_1 and f_2, we see that $f_1 \not\equiv f_2$. Therefore this case is impossible.

The case in which the first letter of f_1 is x^{-1} considered similarly.

Let the first letter of f_1 be y. Then $\alpha_1 = 0$, $\beta_1 > 0$. Since the first letter of f_2 is y, too; therefore, either

$$f_1 \equiv b^{\beta_1}\, a^{\alpha_2}\, b^{\beta_2}\, \ldots\, a^{\gamma}\, x\, y^{-1}\, x^{-1}, \quad \gamma \geq 0,$$

or

$$f_1 \equiv b^{\beta_1}\, a^{\alpha_2}\, b^{\beta_2}\, \ldots\, a^{\gamma}\, x^{-1}\, y^{-1}\, x, \quad \gamma \leq 0.$$

Clearly, in both cases $\beta_1 = 1$. Hence, in the first case

$$f_2 \equiv y\, x\, a^{\delta}\, b^{\beta_l}\, a^{\alpha_{l+1}}\, w, \quad \delta \geq 0,$$

where w is some reduced word. In order that the second letter of f_1 is equal to the second letter of f_2 it is necessary that $\alpha_2 > 0$, i.e.,

$$f_1 \equiv y\, x\, y^{-1}\, x^{-1}\, y\, x\, a^{\alpha_2 - 1}\, b^{\beta_2}\, \ldots\, a^{\gamma}\, x\, y^{-1}\, x^{-1}.$$

In order that the third and fourth letters in the word f_2 are equal to the third and fourth letters of f_1 accordingly, it is necessary that $\delta = 0$, $\beta_l = -1$, $\alpha_{l+1} < 0$, i.e.,

$$f_2 \equiv y\, x\, y^{-1}\, x^{-1}\, y^{-1}\, x\, y\, x\, a^{\alpha_{l+1}-1}\, w,$$

but if we compare the fifth letters f_1 and f_2 then we will see that $f_1 \not\equiv f_2$. In the second case

$$f_2 \equiv y\, x^{-1}\, a^{\delta}\, b^{\beta_l}\, a^{\alpha_{l+1}}\, w, \quad \delta \leq 0.$$

In order that the second letter of f_1 is equal to the second letter of f_2 it is necessary that $\alpha_2 < 0$, i.e.,

$$f_1 \equiv y\, x^{-1}\, y^{-1}\, x\, y\, x^{-1}\, a^{\alpha_2 + 1}\, b^{\beta_2}\, \ldots\, a^{\gamma}\, x^{-1}\, y^{-1}\, x.$$

In order that the third and fourth letters of f_2 are equal to the third and fourth letters of f_1 accordingly; it is necessary that $\delta = 0$, $\beta_l = -1$, $\alpha_{l+1} > 0$, i.e.,

$$f_2 \equiv y\, x^{-1}\, y^{-1}\, x\, y^{-1}\, x^{-1}\, y\, x\, a^{\alpha_{l+1}-1}\, w,$$

but if we compare the fifth letters of f_1 and f_2 then we will see that $f_1 \not\equiv f_2$. Therefore f_1 cannot begin with y.

Arguing similarly, we see that it is not possible for f_1 to begin with y^{-1}. This completes the proof of the lemma and theorem.

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